Kuwait University	Math. 101	December 22, 2009
Dept. of Math. & Comp. Sc.	Second Midterm Exam	Duration 90 Minutes

Calculators and mobile phones are not allowed
Answer the following questions.

1. (**4 Points**) Find an equation for the tangent line to the graph of

$$(x+y)^3 = 1 + x^2y^2$$

at the point whose $x-$coordinate is 0.

2. (**4 Points**) Use a linear approximation to estimate the number

$$N = \frac{\sqrt{2-(1.001)^2}}{1.001}.$$

3. (**4 Points**) A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m^3/min. Find the rate at which the water level is rising when the water is 3 m deep.

4. (**4 Points**) Let $f(x) = x^2 - x + \cos x$. Show that the equation $f'(x) = 0$ has exactly one real solution.

5. (**9 Points**) Let

$$f(x) = \frac{4x^2 - 8x + 8}{x^2}.$$

(a) Show that $f''(x) = \dfrac{16(3-x)}{x^4}$.

(b) Find the vertical and horizontal asymptotes of the graph of f, if any.

(c) Find the intervals on which f is increasing or decreasing, and find the local extrema of f, if any.

(d) Find the intervals on which the graph of f is concave up or concave down, and find the points of inflection, if any.

(e) Sketch the graph of f .

1. $3(x+y)^2(1+y') = 2xy^2 + 2x^2yy'.$ At $x=0:\ y=1$ and $y'=-1$

   An equation for the tangent line at $(0,1)$ is $\boxed{x+y=1.}$

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2. $y = f(x) = \dfrac{\sqrt{2-x^2}}{x} \implies dy = f'(x)\,dx = -\dfrac{2}{x^2\sqrt{2-x^2}}dx.$ When $x=1$ and $dx = 0.001$,

   $dy = -2(0.001) = -0.002$, so $\frac{\sqrt{2-(1.001)^2}}{1.001} = f(1.001) \approx f(1) + dy = 1 - 0.002 = \boxed{0.998.}$

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3. $\dfrac{r}{h} = \dfrac{2}{4} = \dfrac{1}{2} \implies r = \dfrac{1}{2}h \implies V = \dfrac{1}{3}\pi r^2 h = \dfrac{1}{12}\pi h^3 \implies \dfrac{dV}{dt} = \dfrac{1}{4}\pi h^2 \dfrac{dh}{dt} \implies \dfrac{dh}{dt} = \dfrac{4}{\pi h^2}\dfrac{dV}{dt}$

   Substitute $h = 3$ m and $\dfrac{dV}{dt} = 2\ m^3/\min$, we have $\boxed{\dfrac{dh}{dt} = \dfrac{8}{9\pi} \approx 0.28\ m/\min.}$

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4. $f'(x) = 2x - 1 - \sin x$. By I.V. T., the equation $f'(x) = 0$ has a root in $(0,1)$ since f' is continuous on $[0,1]$ and $f'(0)f'(1) < 0$.

   If $f'(x) = 0$ has two real roots a and b with $a < b$, then $f'(a) = 0 = f'(b)$. Since f' is continuous on $[a,b]$ and differentiable on (a,b), Rolle's Theorem implies that there is a number c in (a,b) such that $f''(c) = 0$, but $f''(x) = 2 - \cos x > 0$ for all x, so $f'(x)$ can never be 0. This contradicts $f''(c) = 0$, so $f'(x) = 0$ cannot have two real roots. Hence, it has at most one real root

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5. $f(x) = \dfrac{4x^2 - 8x + 8}{x^2} = 4\left(1 - \dfrac{2}{x} + \dfrac{2}{x^2}\right) \implies f'(x) = \dfrac{8(x-2)}{x^3}$ and $f''(x) = \dfrac{16(-x+3)}{x^4}.$

Interval	$(-\infty,0)$	$(0,2)$	$(2,\infty)$
sign of f'	$+$	$-$	$+$
conclusion $/f$	f is $\nearrow$	f is $\searrow$	f is $\nearrow$

Interval	$(-\infty,0)$	$(0,3)$	$(3,\infty)$
sign of f''	$+$	$+$	$-$
conclusion /graph	CU	CU	CD

$f(2) = 2$ is a local minimum

$\left(3, \frac{20}{9}\right)$ is a point of inflection

$\lim_{x\to 0^\pm} f(x) = \infty \implies \boxed{x = 0 \text{ is V.A.}}$

$\lim_{x\to\pm\infty} f(x) = 4 \implies \boxed{x = 4 \text{ is H.A.}}$